Winding of a two-dimensional Brownian particle in a random environment

1998 J. Phys. A: Math. Gen. 31 9455

(http://iopscience.iop.org/0305-4470/31/47/007)

J. Phys. A: Math. Gen. **31** (1998) 9455–9468. Printed in the UK PII: S0305-4470(98)97452-2

Winding of a two-dimensional Brownian particle in a random environment

K Samokhin†
Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, UK

Received 10 September 1998

Abstract. We study analytically the average probability for a Brownian particle to wind n times around a removed area of finite size in a 2D plane with randomly distributed traps. Such a model describes, for example, the Abrikosov vortex entanglement around a cylindrical cavity in a superconductor with repulsive columnar defects. The problem amounts to the quantum mechanics of a particle moving in a plane with point-like random scatterers pierced by a solenoid. It is shown that at large times t the asymptotic winding angle distribution, which is determined by a 'Lifshitz tail' in the density of states of such a particle, is Gaussian with the scaling variable $x = n/\big(t^{1/4}\ln^{1/2} t\big)$.

1. Introduction

Topological properties of random walks have been a subject of intensive theoretical investigations for many years. A lot of problems of this kind appear quite naturally in studies of the entanglement of polymers (see, e.g. [1]) or Abrikosov vortex lines in superconductors [2]. Perhaps, the most prominent example is the calculation of the winding angle distribution in two dimensions. The winding angle of a planar random walk is the total continuous angle $\theta(t) = 2\pi n(t)$ swept by a Brownian particle around a prescribed point after time t (note that n can be non-integer). It was found by Spitzer [3] that the asymptotic probability to wind n times is given by a Cauchy law:

$$\mathcal{P}(n,t) \sim \frac{1}{1+x^2} \qquad x \sim \frac{n}{\ln t} \qquad \text{at} \quad t \to \infty. \tag{1}$$

The logarithmic scaling can be explained qualitatively if we observe that, for an ideal random walk, the only length scale in the system is the diffusion length $r_D \sim \sqrt{Dt}$, where D is the diffusion coefficient. From dimensional arguments, the increase in winding angle can be written as $\mathrm{d}n \sim \mathrm{d}r_D/r_D \sim \mathrm{d}(\ln t)$, so that the scaling variable is $x \sim n/\ln t$. The result (1) was later confirmed by many authors by employing different techniques [4–7]. Other examples include the winding angle distribution of a self-avoiding random walk [8, 9], the distribution of the algebraic area enclosed by a planar Brownian curve [10, 11], the statistical mechanics of entangled closed polymers [12], etc.

Let us now suppose that our Brownian particle cannot walk freely, but instead can be irreversibly absorbed by the traps located at some randomly distributed points in a plane. It is known that the properties of such a system differ drastically from those of an ideal random walk. For instance, the logarithm of the survival probability is given in dimension

† Permanent address: L D Landau Institute for Theoretical Physics, Kosygina Str. 2, 117940 Moscow, Russia.

d by $\ln P(t) \sim -t^{d/(d+2)}$ [13], while the mean-square displacement is sub-diffusional: $\langle r^2(t)\rangle \sim t^{2/(d+2)}$ [14]. As an example of a real physical system, which is modelled by a two-dimensional random walk with traps, let us mention the statistical mechanics of Abrikosov vortex lines in a three-dimensional (3D) superconductor with columnar defects [15]. If the defects create a repulsive potential for vortices, then the classical vortex partition function obeys the diffusion equation with traps (see equation (5) below), where the time variable t should be replaced by the vertical size of a sample.

It was found in [16] that the asymptotic probability distribution of the winding angle around a prescribed point for a Brownian particle wandering in the presence of random traps obeys a Cauchy law and looks as follows:

$$\frac{\mathcal{P}(n,t)}{P(t)} \sim \frac{1}{1+x^2} \qquad x \sim \frac{n}{\sqrt{t}} \qquad \text{at} \quad t \to \infty \tag{2}$$

where $P(t) = \int_{-\infty}^{+\infty} \mathrm{d}n\, \mathcal{P}(n,t)$ is the survival probability, i.e. the total probability to find a particle anywhere in the plane [13]. The change of the scaling variable from logarithmic to a power law can be understood as follows. In a random medium, another length scale $r_\rho \sim \sqrt{1/\rho}$, which is related to the concentration of traps ρ, comes into play in addition to the diffusion length. It is the presence of this length that changes the asymptotics of the mean-square displacement: $\langle r^2(t)\rangle \sim r_D r_\rho \sim \sqrt{Dt/\rho}$ [14]. Using dimensional arguments, the increase in winding number can be expressed in terms of the two characteristic lengths: $\mathrm{d}n \sim \mathrm{d}r_D/r_\rho$. Therefore, the scaling variable is $x \sim n/\sqrt{\rho D t}$. Qualitatively, a particle is able to survive until time t only if it spends most of its life in a finite region of the plane almost free of traps and thus never wanders too far away from the starting point. Such a restriction obviously results in increasing entanglement, comparing to an ideal random walk.

An obviously unsatisfactory feature of the distribution (2) is that it is so broad that $\langle n^2(t)\rangle$ and all the higher-order moments are infinite. This pathological property, as we shall see, is directly related to the fact that the trajectory of a Brownian particle was allowed to come arbitrarily close to the reference point. The same problem arises in the case of an ideal random walk as well, where the Spitzer distribution (1) also leads to infinite moments. The way out of this problem was pointed out by Rudnick and Hu [17], who suggested to remove a disc of finite radius from the plane, thus preventing the trajectory from coming too close to the reference point. They found that

$$\mathcal{P}(n,t) \sim \exp(-x) \qquad x \sim \frac{n}{\ln t} \qquad \text{at} \quad t \to \infty \tag{3}$$

so that all moments become finite. A similar exponential distribution was obtained in [18] for the winding of a discrete random walk, in which case the lattice constant provides a natural ultraviolet cut-off.

The purpose of the present paper is to find the winding angle distribution for a two-dimensional (2D) random walk *with traps* wandering around an excluded area of finite size. In the case of vortex line entanglement mentioned above, it means that there is, for example, a cylindrical cavity in a superconductor with columnar defects, such that the vortices can wind around it. Our main result is that the average winding probability in this case becomes Gaussian:

$$\frac{\mathcal{P}(n,t)}{P(t)} \sim \exp\left(-x^2\right) \qquad x \sim \frac{n}{t^{1/4}\ln^{1/2} t} \qquad \text{at} \quad t \to \infty. \tag{4}$$

Different boundary conditions at the surface of the excluded disc affect only the numerical factor in the definition of the scaling variable x, but not the general form of the asymptotic distribution function (4).

The paper is organized as follows. In section 2, a direct relationship is established between the winding angle distribution function $\mathcal{P}(n,t)$ and the density of states (DoS) of a quantum particle moving in a 2D plane with random scatterers in the presence of a solenoid. The low-energy behaviour of DoS corresponding to the large-t asymptotics of $\mathcal{P}(n,t)$ is determined by the optimal fluctuations of a random potential, which are described, in the quantum field-theoretical formulation, by instantons. In section 3, an explicit form of the instanton configurations is found for two different kinds of the boundary conditions at the surface of the excluded disc, and the asymptotics of the winding angle distribution functions are calculated with exponential accuracy. Section 4 concludes with a discussion.

2. Field theory formulation

The probability distribution for a continuum random walk to arrive at a point $\boldsymbol{r}$ after time t satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = D\nabla^2 P - U(\boldsymbol{r})P. \tag{5}$$

Here $U(\boldsymbol{r}) = U_0 \sum_i \delta(\boldsymbol{r} - \boldsymbol{R}_i)$ is the random 'potential', which is the probability per unit time and unit volume for a particle to be trapped ($U_0 > 0$). The positions $\boldsymbol{R}_i$ of point-like traps are distributed uniformly in a plane according to the Poisson law with mean density ρ. The restriction that the trajectory of a random walker cannot come too close to the origin is accounted for by the following boundary conditions at the surface of the removed disc of radius R:

$$\begin{aligned} &\text{(A):} \qquad P|_{r=R} = 0 \\ &\text{or (B):} \qquad \left.\frac{\partial P}{\partial r}\right|_{r=R} = 0. \end{aligned} \tag{6}$$

Condition (A), which will be referred to as the inelastic one, describes the situation when the particle gets irreversibly absorbed once it reaches the boundary of the forbidden region, while (B) corresponds to a purely elastic reflection at the boundary.

With the initial condition $P(\boldsymbol{r}, t = 0) = \delta(\boldsymbol{r} - \boldsymbol{r}')$, the formal solution of equation (5) is given by the Wiener path integral formula:

$$P(\boldsymbol{r}, t; \boldsymbol{r}', 0|U) = \int_{r(0)=r'}^{r(t)=r} \mathcal{D}\boldsymbol{r}(\tau)\, \exp\left\{-\int_0^t \mathrm{d}\tau \left(\frac{1}{4D}\dot{\boldsymbol{r}}^2(\tau) + U(\boldsymbol{r}(\tau))\right)\right\}. \tag{7}$$

The probability for a random walk of 'length' t to wind n times around a reference point in a given distribution of traps can be calculated by inserting a δ-function constraint [4] in the expression on the right-hand side of (7), so that

$$\mathcal{P}(n, t|U) = \left\langle \delta\left(n - \frac{1}{2\pi}\int_0^t \mathrm{d}\tau\, \dot{\theta}(\tau)\right)\right\rangle_{P(\boldsymbol{r},t;\boldsymbol{r}',0|U)} \tag{8}$$

where $\theta(t)$ is the angle between the radius-vector $\boldsymbol{r}(t)$ and some fixed direction in the plane. Writing the δ-function as an integral over an auxiliary variable p, we arrive at

$$\begin{aligned} \mathcal{P}(n, t|U) = &\int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{2\pi \mathrm{i} pn} \\ &\times \int_{r(0)=r'}^{r(t)=r} \mathcal{D}\boldsymbol{r}(\tau)\, \exp\left\{-\int_0^t \mathrm{d}\tau \left(\frac{1}{4D}\dot{\boldsymbol{r}}^2(\tau) + U(\boldsymbol{r}(\tau)) + \mathrm{i}p\dot{\theta}(\tau)\right)\right\}. \end{aligned} \tag{9}$$

The path integral on the right-hand side is nothing but the Euclidean Green function $\mathcal{G}_\phi(\boldsymbol{r}, t; \boldsymbol{r}', 0)$ of a particle of unit charge and mass $m = (2D)^{-1}$ moving in a random potential $U(\boldsymbol{r})$ and in the field of a solenoid localized at the origin and carrying a flux $\phi = -2\pi p$. Therefore, the winding probability is given by the following Fourier transform:

$$\mathcal{P}(n, t; \boldsymbol{r}, \boldsymbol{r}') = \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \mathrm{e}^{-\mathrm{i}\phi n} \mathcal{G}_\phi(\boldsymbol{r}, t; \boldsymbol{r}', 0). \tag{10}$$

The Green function on the right-hand side has the following explicit form:

$$\mathcal{G}_\phi(\boldsymbol{r}, t; \boldsymbol{r}', 0) = \sum_i \psi_i(\boldsymbol{r})\, \psi_i(\boldsymbol{r}')\, \mathrm{e}^{-E_i t} \tag{11}$$

where $\psi_i(\boldsymbol{r})$ and E_i are the eigenfunctions and eigenvalues of the Hamiltonian

$$H = D(-\mathrm{i}\nabla - \boldsymbol{A}(\boldsymbol{r}))^2 + U(\boldsymbol{r}). \tag{12}$$

Here $A_\theta = \phi/2\pi r$ is the vector potential created by the solenoid. The functions $\psi_i(\boldsymbol{r})$ must satisfy the boundary conditions (6). Note that, since the random potential $U(\boldsymbol{r})$ is positive, all the eigenvalues $E_i > 0$.

If we assume that the trajectory is closed (i.e. $\boldsymbol{r} = \boldsymbol{r}'$) and the starting point $\boldsymbol{r}$ is not fixed, then, after integration over $\boldsymbol{r}$, divided by the system volume Ω, we obtain the following expression for the average probability:

$$\mathcal{P}(n, t) \equiv \langle \mathcal{P}(n, t|U) \rangle_U = \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\infty} \mathrm{d}E\, \mathrm{e}^{-\mathrm{i}\phi n}\, \mathrm{e}^{-Et} N(E, \phi) \tag{13}$$

where $N(E, \phi) = (1/\Omega)\big\langle \sum_i \delta(E - E_i(\phi)) \big\rangle_U$ is the average density of states. The asymptotic behaviour of $\mathcal{P}(n, t)$ at large t is thus determined by the asymptotics of $N(E, \phi)$ at small E, which is called the 'Lifshitz tail' [19].

From the analysis of a random walk with traps but without solenoid it is known that at $E \to 0$ the main contribution to the density of states comes from the large regions in real space which are free of traps. The probability to find such a region of area A is exponentially small: $p(A) \sim \mathrm{e}^{-\rho A}$. On another hand, the ground state energy of a particle in a 2D potential well of radius R is given by $E(R) \sim DR^{-2} \sim DA^{-1}$. Therefore, $A(E) \sim D/E$, and the density of states is $N(E) \sim p(A(E)) \sim \exp(-\text{constant} \times \rho D/E)$. More formally, such exponentially small tails of the density of states correspond to the contribution of instantons, which are spatially localized solutions of the saddle-point equations in the functional–integral representation of the problem [20, 21]. Below, following a general procedure of [22, 23], we find the instanton solutions in the presence of a solenoid.

The density of states is $N(E) = -(1/\pi\Omega) \int \mathrm{d}^2 r\, \mathrm{Im}\, G^R(\boldsymbol{r}, \boldsymbol{r}; E)$, where $G^R(\boldsymbol{r}, \boldsymbol{r}'; E) = \big\langle \boldsymbol{r} \big| (E - H + \mathrm{i}0)^{-1} \big| \boldsymbol{r}' \big\rangle$ is the retarded Green function of the Schrödinger equation with the Hamiltonian (12), which can be calculated by standard means of the quantum field theory. Using the replica trick, the non-averaged retarded Green function is written in the following form (in the limit $n \to 0$):

$$G^R(\boldsymbol{r}, \boldsymbol{r}'; E) = -\mathrm{i} \lim_{\eta \to +0} \int \mathcal{D}^2 \boldsymbol{\varphi}(\boldsymbol{r})\, \exp\left(\mathrm{i} \int \mathrm{d}^2 r\, \bar{\boldsymbol{\varphi}}(E - H + \mathrm{i}\eta) \boldsymbol{\varphi} \right) \varphi^1(\boldsymbol{r})\, \bar{\varphi}^1(\boldsymbol{r}') \tag{14}$$

where $\boldsymbol{\varphi} = \big(\varphi^1, \ldots, \varphi^n\big)$ is an n-component Bose field and $\mathcal{D}^2 \boldsymbol{\varphi} = \prod_{a=1}^n \mathcal{D}\big(\mathrm{Re}\, \varphi^a\big) \times \mathcal{D}\big(\mathrm{Im}\, \varphi^a\big)/\pi$. The functional integral is convergent due to the presence of the term with η. It is easy to see that the boundary conditions (6) for the Green function are automatically satisfied if we impose the corresponding conditions on the fields $\varphi^a(\boldsymbol{r})$. After disorder averaging, the exponent on the right-hand side of equation (14) takes the form

$$\mathrm{i}S[\boldsymbol{\varphi}] = \int \mathrm{d}^2 r\, \big\{ \mathrm{i}\bar{\boldsymbol{\varphi}}\big(E - D(-\mathrm{i}\nabla - \boldsymbol{A})^2 + \mathrm{i}\eta\big)\boldsymbol{\varphi} - \rho\big(1 - \mathrm{e}^{-\mathrm{i}U_0 \bar{\boldsymbol{\varphi}}\boldsymbol{\varphi}}\big) \big\}. \tag{15}$$

It is convenient to write the fields $\boldsymbol{\varphi}$ as $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \mathrm{i}\boldsymbol{\varphi}_2$, $\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi}_1 - \mathrm{i}\boldsymbol{\varphi}_2$, where $\boldsymbol{\varphi}_{1,2} = \left(\varphi^1_{1,2}, \ldots, \varphi^n_{1,2}\right)$ are real on the initial functional integration contour. The saddle points of the action satisfy the following equation:

$$D(-\mathrm{i}\nabla - \boldsymbol{A}(\boldsymbol{r}))^2\boldsymbol{\varphi}_{1,2} + E_\mathrm{c}\mathrm{e}^{-\mathrm{i}U_0(\boldsymbol{\varphi}_1^2+\boldsymbol{\varphi}_2^2)}\boldsymbol{\varphi}_{1,2} = E\boldsymbol{\varphi}_{1,2} \tag{16}$$

where $E_\mathrm{c} = \rho U_0$ is the mean value of the random potential. Except from an obvious solution $\boldsymbol{\varphi}_{1,2} = 0$, equation (16) also has non-trivial solutions in the complex plane of $\boldsymbol{\varphi}_{1,2}$. To find their explicit form, it is convenient to rotate the functional integration contour: $\boldsymbol{\varphi}_{1,2} \to \mathrm{e}^{-\mathrm{i}\pi/4}\tilde{\boldsymbol{\varphi}}_{1,2}$, which makes equation (16) real. Due to the rotational symmetry of the action (15) in the n-dimensional replica space, we are able to seek a solution of the saddle-point equations in the following form:

$$\tilde{\varphi}_{1,a}(\boldsymbol{r}) = \frac{1}{\sqrt{2}}\varphi(\boldsymbol{r})\, e_{1,a} \qquad \tilde{\varphi}_{2,a}(\boldsymbol{r}) = \frac{1}{\sqrt{2}}\varphi(\boldsymbol{r})\, e_{2,a} \tag{17}$$

where $\boldsymbol{e}_{1,2}$ are arbitrary n-component unit vectors. From (16) and (17) we obtain the following equation for the function $\varphi(\boldsymbol{r})$ which is assumed to be rotationally invariant in real space (i.e. $\varphi(\boldsymbol{r}) = \varphi(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ with $m = 0$):

$$-D\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)\varphi + D\frac{\nu^2}{r^2}\varphi + E_\mathrm{c}\mathrm{e}^{-U_0\varphi^2}\varphi = E\varphi \tag{18}$$

where $\nu = |\phi|/2\pi$.

At $E < 0$ equation (18) only has the solution $\varphi = 0$. However, at $0 < E < E_\mathrm{c}$ a non-zero solution $\varphi = \varphi_{\mathrm{inst}}(r)$ (instanton) also exists. It is easy to see that a non-trivial saddle point contributes to the imaginary part of the Green function. Indeed, the rotation of the functional integration contour mentioned above cancels the overall factor i in equation (14) and also changes the exponent: $\mathrm{i}S \to -S$, where

$$S[\tilde{\boldsymbol{\varphi}}_{1,2}] = \int \mathrm{d}^2 r\,\left\{\tilde{\varphi}_i^a\left(-E + D(-\mathrm{i}\nabla - \boldsymbol{A})^2\right)\tilde{\varphi}_i^a + \rho\left(1 - \mathrm{e}^{-U_0\tilde{\varphi}_i^a\tilde{\varphi}_i^a}\right)\right\} \tag{19}$$

($i = 1, 2$). At $0 < E < E_\mathrm{c}$ this action describes a metastable field theory with a non-zero saddle point at the real axis in the complex plane of $\tilde{\boldsymbol{\varphi}}_{1,2}$. In order to restore convergence, one has once again to rotate the functional integration contour leaving this saddle point by $\mathrm{e}^{\mathrm{i}\pi/2}$. Such a procedure generates the required factor i in front of the functional integral. Finally, at $0 < E \ll E_\mathrm{c}$ the density of states is exponentially small:

$$N(E,\phi) \sim \mathrm{e}^{-S_{\mathrm{inst}}(E,\phi)} \tag{20}$$

where the instanton action is

$$S_{\mathrm{inst}} = \int \mathrm{d}^2 r\,\left\{\varphi_{\mathrm{inst}}\left(-E + D(-\mathrm{i}\nabla - \boldsymbol{A})^2\right)\varphi_{\mathrm{inst}} + \rho\left(1 - \mathrm{e}^{-U_0\varphi^2_{\mathrm{inst}}}\right)\right\}. \tag{21}$$

The pre-exponential factor can be calculated by expanding the action (19) around the instanton solution, followed by the Gaussian integration over all modes with non-zero eigenvalues [22, 23]. We, however, shall not proceed further in this direction. It is just worth noticing that, within the exponential accuracy, when the time dependence of the Green function at large t is determined by the instanton contributions, its coordinate dependence contained in the pre-exponential factor (see equation (14)) is not important. Therefore, we do not expect the results in the asymptotic regime to depend on $\boldsymbol{r}$ and also on whether we consider a closed or an open trajectory.

Let us introduce the dimensionless variables

$$r = \xi x \qquad \varphi_{\mathrm{inst}}(x) = U_0^{-1/2} f(x).$$

Here $\xi = \sqrt{D/E}$ is a characteristic scale of the problem, which is nothing but the typical length of diffusion in time $t = E^{-1}$. Equation (18) can then be written as

$$-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)f + \frac{\nu^2}{x^2}f + \alpha^2 \mathrm{e}^{-f^2} f = f \tag{22}$$

where $\alpha^2 = E_\mathrm{c}/E$. The solution of this equation must be finite everywhere and satisfy the following boundary conditions at the surface of the removed disc:

$$f|_{x=X} = 0 \tag{23}$$

in the inelastic case, or

$$\left.\frac{\partial f}{\partial x}\right|_{x=X} = 0 \tag{24}$$

in the elastic case. Here $X = R/\xi \ll 1$ in the limit $E \ll E_\mathrm{c}$.

In the derivation of equation (22), we introduced from the beginning a continuum random walk described by the diffusion equation (5). Alternatively, one might start, following Lubensky [21], from a slightly different discrete model, in which a Brownian particle walks on a square lattice, whose sites are occupied by traps with probability $p < 1$. Traps annihilate the particle once it reaches them. It can be shown that the calculation of the winding distribution for such a system reduces to finding the instanton configurations in some effective field theory in the presence of a solenoid. In the continuum limit, this field theory is characterized by an action similar to (19), but with a different 'potential energy':

$$\rho\left(1 - \mathrm{e}^{-U_0\tilde{\varphi}_i^a\tilde{\varphi}_i^a}\right) \to -\ln\left\{1 - p + p\mathrm{e}^{-\tilde{\varphi}_i^a\tilde{\varphi}_i^a}\right\}. \tag{25}$$

We shall see in the next section that the difference between the two field theories is actually not essential, so that both of them can be dealt with in a similar fashion.

3. Instanton solutions and Lifshitz tails

Due to the complexity of the saddle-point equation (22), it is possible to find only an approximate solution. To do that, we replace the 'potential' $V(f) = \alpha^2\mathrm{e}^{-f^2}$ in the nonlinear equation (22) by the following piecewise constant effective potential:

$$V_{\mathrm{eff}}(f) = \begin{cases} \alpha^2 & \text{at} \quad f < 1 \\ 0 & \text{at} \quad f > 1. \end{cases} \tag{26}$$

It is assumed that $E \ll E_\mathrm{c}$, i.e. $\alpha \gg 1$. The problem thus amounts to solving a set of *linear* Schrödinger-like equations complemented by the requirement of continuity of both the solution and its derivative at certain matching points x_i (which are the boundaries of the effective potential well), whose positions are to be determined from the conditions $f(x_i - 0) = f(x_i + 0) = 1$.

The same trick also works for the lattice model mentioned at the end of the previous section. In this case, the dimensionless field is $f = \varphi$ and we obtain from (25) the following 'potential' in the nonlinear saddle-point equation:

$$V(f) = \frac{p}{E}\frac{\mathrm{e}^{-f^2}}{1 - p + p\mathrm{e}^{-f^2}}.$$

It is easy to see that $V(f)$ can be replaced by the effective potential (26) with $\alpha^2 = p/E \gg 1$. Therefore, all subsequent calculations and results are the same for both models. Let us start from the case of an absorbing wall.

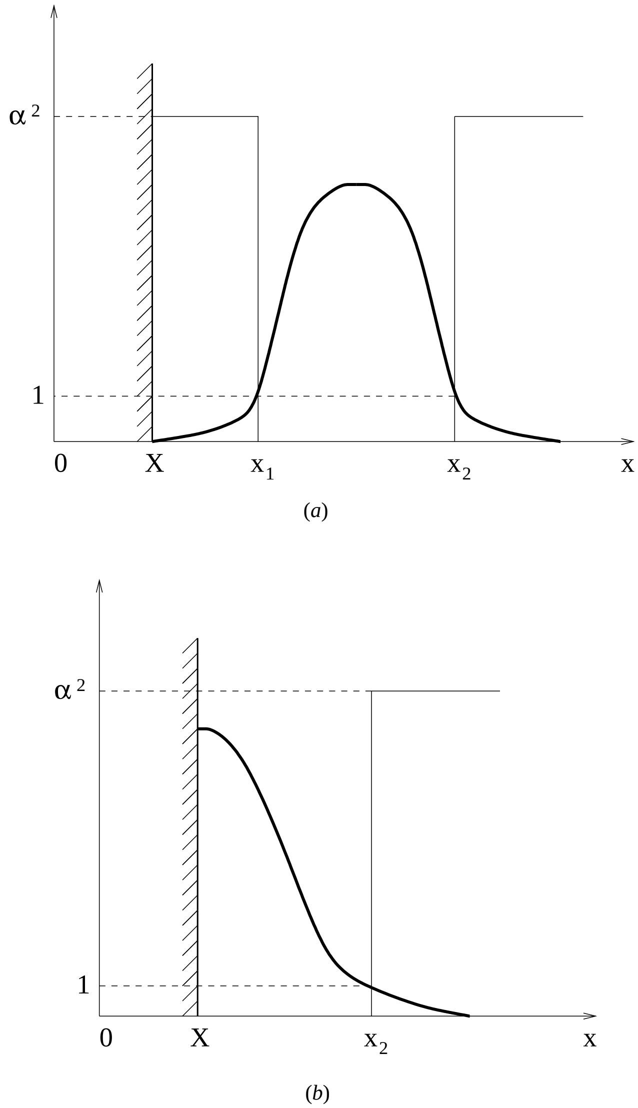

Figure 1. The effective potential $V_{\mathrm{eff}}(f(x))$ (thin line) and the instanton solution $f(x)$ (thick curve) as functions of $x = r/\xi$ in the case of (a) an absorbing boundary and (b) a reflecting boundary. X is the position of the boundary and $\alpha^2 = E_c/E \gg 1$.

3.1. Absorbing wall

As seen from equation (23), the condition $f(x) > 1$ cannot be satisfied for x sufficiently close to X. On another hand, $f(x)$ must vanish as $x \to \infty$. It is clear, therefore, that one needs at least two matching points in this case, so that the solution is a piecewise continuous function (see figure 1(a)):

$$f(x) = \begin{cases} f_1(x) & \text{at} \quad X < x < x_1 \\ f_2(x) & \text{at} \quad x_1 < x < x_2 \\ f_3(x) & \text{at} \quad x_2 < x \end{cases} \tag{27}$$

where the functions $f_i(x)$ obey the following linear equations:

$$
\begin{aligned}
&-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)f_{1,3}+\frac{\nu^2}{x^2}f_{1,3}+\alpha^2 f_{1,3}=f_{1,3}\\
&-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)f_2+\frac{\nu^2}{x^2}f_2=f_2.
\end{aligned}
\tag{28}
$$

A general solution of these equations, which tends to zero at $x\to\infty$, looks as follows:

$$
\begin{aligned}
f_1 &= C_1^{(1)}I_\nu(\alpha x)+C_1^{(2)}K_\nu(\alpha x)\\
f_2 &= C_2^{(1)}J_\nu(x)+C_2^{(2)}Y_\nu(x)\\
f_3 &= C_3K_\nu(\alpha x)
\end{aligned}
\tag{29}
$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively, and $I_\nu(x)$ and $K_\nu(x)$ are the Bessel functions of imaginary argument. The boundary conditions read

$$
\begin{aligned}
&f_1(x_1)=f_2(x_1)=1 \qquad f_2(x_2)=f_3(x_2)=1\\
&f_1'(x_1)=f_2'(x_1) \qquad f_2'(x_2)=f_3'(x_2)\\
&f_1(X)=0.
\end{aligned}
\tag{30}
$$

Substituting (29) in (30), we obtain a system of seven transcendent equations to determine $C_1^{(1,2)}$, $C_2^{(1,2)}$, C_3, x_1 and x_2. After changing notations $C_2^{(1,2)}=\alpha A_{1,2}$, the equations for $A_{1,2}$ and $x_{1,2}$ take the form

$$
\begin{aligned}
&A_1J_\nu(x_1)+A_2Y_\nu(x_1)=\alpha^{-1}\\
&A_1J_\nu(x_2)+A_2Y_\nu(x_2)=\alpha^{-1}\\
&A_1J_\nu'(x_1)+A_2Y_\nu'(x_1)=F_\nu(x_1,X)\\
&A_1J_\nu'(x_2)+A_2Y_\nu'(x_2)=\frac{K_\nu'(\alpha x_2)}{K_\nu(\alpha x_2)}
\end{aligned}
\tag{31}
$$

where

$$
F_\nu(x_1,X)=\frac{K_\nu(\alpha X)\,I_\nu'(\alpha x_1)-I_\nu(\alpha X)\,K_\nu'(\alpha x_1)}{K_\nu(\alpha X)\,I_\nu(\alpha x_1)-I_\nu(\alpha X)\,K_\nu(\alpha x_1)}
\tag{32}
$$

($x_1>X$). In the limit $\alpha\gg 1$ the right-hand sides of the first two equations vanish. If to assume that $x_2\sim 1$ (we shall see below that this assumption is self-consistent), then at $\alpha\gg 1$ the right-hand side of the fourth equation in (31) tends to -1. Excluding $A_{1,2}$ from equations (31) and using the expression for the Wronskian of the Bessel functions $W\{J_\nu(x),Y_\nu(x)\}=2/(\pi x)$, we end up with two equations for $x_{1,2}$:

$$
\begin{aligned}
&x_1F_\nu(x_1,X)\,J_\nu(x_1)=-x_2\,J_\nu(x_2)\\
&x_1F_\nu(x_1,X)\,Y_\nu(x_1)=-x_2\,Y_\nu(x_2).
\end{aligned}
\tag{33}
$$

These equations are valid for $E\ll E_{\mathrm{c}}$ and arbitrary ν and X. Going back to the dimensional variables, it is easy to see from (21) that the instanton action is proportional to the area of the effective potential well:

$$
S_{\mathrm{inst}}=2\pi\rho\int_{r_1}^{r_2}\mathrm{d}r\,r=\pi\rho\xi^2\big(x_2^2(\nu)-x_1^2(\nu)\big).
\tag{34}
$$

The behaviour of $x_{1,2}$ and, therefore, S_{inst} at $\nu\to 0$ turns out to be qualitatively different depending on whether X is zero or not.

$\boldsymbol{X} = \mathbf{0}$. Let us first consider the case when the removed disc shrinks to a point and briefly recall how the result (2) was obtained [16]. The function F looks as follows:

$$F_\nu(x_1, X = 0) = \frac{I'_\nu(\alpha x_1)}{I_\nu(\alpha x_1)} = \frac{\nu}{\alpha x_1} + \frac{I_{\nu+1}(\alpha x_1)}{I_\nu(\alpha x_1)}. \tag{35}$$

In the absence of solenoid (i.e. at $\nu = 0$), equations (33) read

$$x_1 \frac{I_1(\alpha x_1)}{I_0(\alpha x_1)} = -x_2 \frac{J_0(x_2)}{J_0(x_1)} \qquad x_1 \frac{I_1(\alpha x_1)}{I_0(\alpha x_1)} = -x_2 \frac{Y_0(x_2)}{Y_0(x_1)}.$$

The solution of these equations is $x_{1,0} = 0$, $x_{2,0} = a$, where $a \approx 2.405$ is the first zero of the function $J_0(x)$. After substitution in (34), the Lifshitz result [19] is recovered.

At α fixed and $\nu \to 0$ we can use a perturbative expansion in powers of ν and seek a solution of equations (33) in the form $x_1 = \delta x_1(\nu)$, $x_2 = a + \delta x_2(\nu)$, where $\alpha\delta x_1 \to 0$. Then, using the identities [24]

$$J_\nu(x) = J_0(x) + \tfrac{1}{2}\pi\nu Y_0(x) \qquad Y_\nu(x) = Y_0(x) - \tfrac{1}{2}\pi\nu J_0(x) \qquad \text{at} \quad \nu \to 0 \tag{36}$$

and also the property $J'_0(x) = -J_1(x)$, equations (33) and (35) can be reduced, in the leading order in $\delta x_{1,2}$ and ν, to the following form:

$$\frac{1}{\alpha}\frac{1}{\Gamma(\nu)}\left(\tfrac{1}{2}\alpha\delta x_1\right)^{\nu} = a\, J_1(a)\,\delta x_2 - \tfrac{1}{2}\pi a\, Y_0(a)\,\nu$$
$$\frac{1}{\alpha}\frac{\Gamma(\nu+1)}{\pi}\left(\tfrac{1}{2}\alpha\delta x_1\right)^{-\nu} = a\, Y_0(a)$$

where $\Gamma(x)$ is the gamma function. Therefore, we have

$$\begin{aligned} x_1 &= \frac{2}{\alpha}\left(\frac{\pi a\, Y_0(a)\,\alpha}{\Gamma(\nu+1)}\right)^{-1/\nu} \simeq \frac{2}{\alpha}(\pi a\, Y_0(a)\,\alpha)^{-1/\nu} \\ x_2 &= a + \left(\frac{\pi Y_0(a)}{2J_1(a)} + \frac{1}{\pi a^2 J_1(a)\, Y_0(a)\,\alpha^2}\right)\nu \simeq a + \frac{\pi Y_0(a)}{2J_1(a)}\nu. \end{aligned} \tag{37}$$

Since at $\nu \to 0$, x_1 goes to zero faster than δx_2, its contribution to the instanton action can be neglected. From (34), we then obtain [16]

$$N(E, \phi) \sim \mathrm{e}^{-S_{\mathrm{inst}}(E,\phi)} = \exp\left(-\frac{\pi\rho D a^2(1 + b_0|\phi|)}{E}\right) \tag{38}$$

where $b_0 = Y_0(a)/2aJ_1(a) \approx 0.204$. Substituting (38) in (13) and calculating the integrals by the steepest-descent method, we end up with equation (2).

It is seen from the above analysis that the pathologically broad distribution function (2) appears as a result of a non-analytical, linear in $|\phi|$, behaviour of the density of states at small ϕ. This non-analiticity, in turn, is related to the fact that one cannot regard the solenoid field as a small perturbation due to $1/x^2$-singularity at small x. The singularity becomes important as the inner radius of the effective potential well goes to zero at $\nu \to 0$.

$\boldsymbol{X} \neq \mathbf{0}$. In the subsequent analysis we assume that the radius R of the removed disc is small, i.e. $\alpha X \ll 1$. It means that $R \ll R_{\mathrm{c}}$, where $R_{\mathrm{c}} = \sqrt{D/E_{\mathrm{c}}}$. We also assume that x_1 is sufficiently close to X so that $\alpha x_1 \ll 1$. In this limit the equations can be considerably simplified. Substituting in equation (32) the asymptotic expansions of the Bessel functions at small arguments, we obtain

$$F_\nu(x_1, X) = \frac{\nu}{\alpha x_1}\frac{(x_1/X)^\nu + (x_1/X)^{-\nu}}{(x_1/X)^\nu - (x_1/X)^{-\nu}}. \tag{39}$$

It is obvious from this expression that the Taylor expansion of F_ν contains only even powers of ν.

Let us first consider the case of $\nu = 0$. We seek a solution of equations (33) in the form $x_{1,0} = X + \Delta x_1$, $x_{2,0} = a + \Delta x_2$, where $\Delta x_1 \ll X$, $\Delta x_2 \ll a$ at $E \to 0$. It follows from (39) that

$$F_0(x_{1,0}, X) = \frac{1}{\alpha \Delta x_1}.$$

Substituting this expression in equations (33) and using the asymptotics $Y_0(x) \simeq (2/\pi) \ln x$ at small x, we have

$$\begin{aligned} x_{1,0} &= X + \frac{2}{\pi a\, Y_0(a)} \frac{X}{\alpha} \ln \frac{1}{X} \\ x_{2,0} &= a + \frac{\pi Y_0(a)}{2 J_1(a)} \ln^{-1} \frac{1}{X}. \end{aligned} \tag{40}$$

The difference between $x_{1,0}$ and $X \sim \sqrt{E}$ vanishes at $E \to 0$ faster than X itself (as $E|\ln E|$), so that our initial assumption that $x_1 - X \ll X$ is self-consistent.

At $\nu \neq 0$, we seek a solution of equations (33) in the following form: $x_i = x_{i,0} + \delta x_i(\nu, X)$, where $\delta x_i \ll \Delta x_i$ $(i = 1, 2)$. The leading terms in the expansion of F_ν in powers of δx_1 and ν can be found from (39):

$$F_\nu(x_1, X) = \frac{1}{\alpha \Delta x_1} - \frac{1}{\alpha (\Delta x_1)^2} \delta x_1 + \frac{\Delta x_1}{3\alpha X^2} \nu^2.$$

After substitution of this expression and the expansions of the Bessel functions (A1) from the appendix, equations (33) can be linearized with respect to $\delta x_{1,2}$:

$$\begin{aligned} L_{11}\delta x_1 + L_{12}\delta x_2 &= M_1 \nu^2 \\ L_{21}\delta x_1 + L_{22}\delta x_2 &= M_2 \nu^2 \end{aligned} \tag{41}$$

where

$$\begin{aligned} L_{11} &= \frac{x_{1,0} J_1(x_{1,0}) - J_0(x_{1,0})}{\alpha \Delta x_1} + \frac{x_{1,0} J_0(x_{1,0})}{\alpha (\Delta x_1)^2} \simeq \frac{\pi^2 a^2 Y_0^2(a)}{4} \frac{\alpha}{X} \ln^{-2} \frac{1}{X} \\ L_{12} &= x_{2,0} J_1(x_{2,0}) - J_0(x_{2,0}) \simeq a J_1(a) \\ L_{21} &= \frac{x_{1,0} Y_1(x_{1,0}) - Y_0(x_{1,0})}{\alpha \Delta x_1} + \frac{x_{1,0} Y_0(x_{1,0})}{\alpha (\Delta x_1)^2} \simeq -\frac{\pi a^2 Y_0^2(a)}{2} \frac{\alpha}{X} \ln^{-1} \frac{1}{X} \\ L_{22} &= x_{2,0} Y_1(x_{2,0}) - Y_0(x_{2,0}) \simeq a Y_1(a) - Y_0(a) \end{aligned}$$

and

$$\begin{aligned} M_1 &= \frac{x_{1,0}\, j_2(x_{1,0})}{\alpha \Delta x_1} + \frac{x_{1,0} J_0(x_{1,0}) \Delta x_1}{3\alpha X^2} + x_{2,0}\, j_2(x_{2,0}) \simeq \frac{\pi a\, Y_0(a)}{4} \ln \frac{1}{X} \\ M_2 &= \frac{x_{1,0}\, y_2(x_{1,0})}{\alpha \Delta x_1} + \frac{x_{1,0} Y_0(x_{1,0}) \Delta x_1}{3\alpha X^2} + x_{2,0}\, y_2(x_{2,0}) \simeq -\frac{a Y_0(a)}{6} \ln^2 \frac{1}{X}. \end{aligned}$$

Here we singled out the leading contributions to L_{ij} and M_i at $E \to 0$. Note that all the terms linear in ν cancelled each other on the right-hand sides of equations (41). Solving the linear equations, we finally obtain

$$\begin{aligned} \delta x_1 &= \frac{1}{3\pi a\, Y_0(a)} \nu^2 \frac{X}{\alpha} \ln^3 \frac{1}{X} \\ \delta x_2 &= \frac{\pi Y_0(a)}{6 J_1(a)} \nu^2 \ln \frac{1}{X}. \end{aligned} \tag{42}$$

Substituting (42) in (34), we see that the flux dependence of the instanton action is determined, in the leading order in E, by the deviation of the outer radius $x_2(\nu)$ from $x_{2,0}$. Going back to the dimensional variables, we obtain

$$S_{\mathrm{inst},A}(E,\phi) = \pi\rho\xi^2 a^2\left(1+\frac{2}{a}\delta x_2(\nu)\right) = \frac{\pi\rho D a^2}{E}\left(1+b_1\phi^2\ln\frac{E_0}{E}\right) \tag{43}$$

where $b_1 = Y_0(a)/24\pi a J_1(a) \approx 0.005$ and $E_0 = D/R^2$.

After substitution of (43) in (20) and (13), we arrive at

$$\mathcal{P}_A(n,t) \sim \int_{-\infty}^{\infty}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}E\,\mathrm{e}^{-\mathrm{i}\phi n}\mathrm{e}^{-Et}\exp\left\{-\frac{\pi\rho D a^2}{E}\left(1+b_1\phi^2\ln\frac{E_0}{E}\right)\right\}.$$

At large t the integral over E can be calculated by the method of steepest descent, resulting in

$$\begin{aligned}\mathcal{P}_A(n,t) &\sim \int_{-\infty}^{\infty}\mathrm{d}\phi\,\mathrm{e}^{-\mathrm{i}\phi n}\exp\left\{-2\sqrt{\pi\rho D a^2 t}\left(1+\frac{b_1}{4}\phi^2\ln\frac{E_0^2 t}{\pi\rho D a^2}\right)\right\}\\ &\sim \exp\left(-2\sqrt{\pi\rho D a^2 t}\right)\exp\left\{-\frac{n^2}{2ab_1}\frac{1}{\sqrt{\pi\rho D t}}\ln^{-1}\frac{E_0^2 t}{\pi\rho D a^2}\right\}.\end{aligned} \tag{44}$$

The first exponential factor on the right-hand side of (44) represents the asymptotic probability $P(t)$ for a particle without solenoid to survive after time t and coincides with the result of Balagurov and Vaks [13]. The second factor can thus be interpreted as the conditional probability for a particle which has survived until time t to wind n times around the removed disc.

3.2. Reflecting wall

In this case the calculations are simpler, since we are able to introduce just one matching point x_2, i.e. put $x_1 = X$ from the beginning, so that

$$S_{\mathrm{inst}} = \pi\rho\xi^2 x_2^2(\nu) - R^2. \tag{45}$$

The solution of the linearized saddle-point equation with the effective potential (26) looks as follows (see figure 1(*b*)):

$$f(x) = \begin{cases} f_2(x) & \text{at} \quad X < x < x_2 \\ f_3(x) & \text{at} \quad x_2 < x \end{cases} \tag{46}$$

where the functions $f_{2,3}(x)$ obey the equations (28) complemented by the boundary conditions

$$\begin{aligned} &f_2(x_2) = f_3(x_2) = 1\\ &f_2'(x_2) = f_3'(x_2)\\ &f_2'(X) = 0. \end{aligned} \tag{47}$$

Substituting the solutions (29) in (47), we end up, in the limit $\alpha \to \infty$ and $x_2 \sim 1$, with the following equation for x_2:

$$\frac{J_\nu(x_2)}{Y_\nu(x_2)} = \frac{J_\nu'(X)}{Y_\nu'(X)}. \tag{48}$$

In the absence of solenoid, we obtain

$$x_{2,0} = a + \frac{\pi Y_0(a)}{4J_1(a)}X^2 \qquad \text{at} \quad X \ll 1. \tag{49}$$

At $\nu \neq 0$, we seek a solution of equation (48) in the form $x_2 = x_{2,0} + \delta x_2(\nu)$. Substituting the expansions (A1) and (A2) in equation (48), we find that the terms of the first order in ν cancel out, and the leading contribution to δx_2 at $X \ll 1$ looks as follows:

$$\delta x_2 = \frac{Y_0(x_{2,0})\, j_2'(X) + j_2(x_{2,0})\, Y_1(X) - J_0(x_{2,0})\, y_2'(X) - y_2(x_{2,0})\, J_1(X)}{J_1(x_{2,0})\, Y_1(X) - Y_1(x_{2,0})\, J_1(X)} \nu^2$$
$$\simeq \frac{\pi Y_0(a)}{2J_1(a)} \nu^2 \ln \frac{1}{X}. \tag{50}$$

Substitution of this expression in (45) results in

$$S_{\text{inst},B} = \frac{\pi \rho D a^2}{E}\left(1 + b_2 \phi^2 \ln \frac{E_0}{E}\right) \tag{51}$$

where $b_2 = Y_0(a)/8\pi a J_1(a) = 3b_1 \approx 0.015$. Comparing this result with (43), we see that the flux-dependent correction to the instanton action in the presence of a solenoid is not sensitive (up to a numerical coefficient) to the boundary conditions at the surface of the excluded disc. Proceeding as in the previous subsection, we finally obtain

$$\mathcal{P}_B(n,t) \sim \exp\left(-2\sqrt{\pi \rho D a^2 t}\right) \exp\left\{-\frac{n^2}{2ab_2} \frac{1}{\sqrt{\pi \rho D t}} \ln^{-1} \frac{E_0^2 t}{\pi \rho D a^2}\right\}. \tag{52}$$

To conclude this section, it is worth noticing that the flux dependence of the low-energy asymptotics of DoS can be estimated from rather simple qualitative considerations, following the heuristic argumentation of Lifshitz [19] extended to the case of a non-zero flux. Indeed, it was found in subsection 3.1 that the difference between the inner matching point x_1 and X can be safely neglected. Therefore, for both types of boundary conditions the problem of finding the low-energy tail of DoS amounts to the calculation of the ground state energy E of a quantum particle confined to move in a trap-free region, which is modelled by a potential well with infinitely high walls, having the shape of a coaxial ring with inner and outer radii $R \neq 0$ and $r_2 \simeq a\xi \gg R$, respectively. Under these conditions, one should expect a perturbation theory with respect to ϕ to work well, since the $1/r^2$-singularity of the solenoid field is now cut off by the finite radius R. The flux-dependent correction to E then reads

$$\delta E(\phi) = \frac{\int_R^{a\xi} r\,\mathrm{d}r \left(D\phi^2 / 4\pi^2 r^2\right) \psi_0^2(r)}{\int_R^{a\xi} r\,\mathrm{d}r\, \psi_0^2(r)} \tag{53}$$

where $\psi_0(r) \sim J_0(r/\xi)$ is the ground state wavefunction in the absence of the solenoid. Calculating the integrals, we obtain: $\delta E(\phi) \sim D\phi^2 \xi^{-2} \ln(\xi/R) \sim \phi^2 E \ln(E_0/E)$. To keep the ground state energy fixed, one has to compensate the correction (53) by increasing the area of a trap-free region: $\delta A(E,\phi) \sim (D/E)(\delta E(\phi)/E)$. Therefore, the DoS at a fixed energy decreases: $N(E,\phi) \sim \exp\{-\rho(A(E) + \delta A(E,\phi))\} \sim N(E,\phi = 0) \exp\left\{-\text{constant} \times (\rho D/E)\phi^2 \ln(E_0/E)\right\}$, which agrees with the expressions (43) and (51) obtained by more rigorous analysis.

4. Conclusions and discussion

Summarizing, for both kinds of boundary conditions the average winding probability is given by the following expression:

$$\frac{\mathcal{P}(n,t)}{P(t)} \sim \exp\left\{-c\left(\frac{n}{n_0}\right)^2 \left(\frac{t_0}{t}\right)^{1/2} \ln^{-1} \frac{t}{n_0^4 t_0}\right\} \qquad \text{at} \quad t \to \infty \tag{54}$$

where $P(t) = \int_{-\infty}^{\infty} \mathrm{d}n\, \mathcal{P}(n,t)$ is the survival probability, and $c_A = 1/2ab_1 \approx 41.6$, $c_B = 1/2ab_2 \approx 13.9$. The characteristic scales of the winding number and time are

$$n_0 = \left(\pi \rho R^2\right)^{1/4} = N_R^{1/4} \qquad t_0 = E_0^{-1} = \frac{R^2}{D} \tag{55}$$

with a simple physical meaning: N_R is the number of traps per area of the excluded disc, and t_0 is the time of diffusion at distance R. We see that the presence of a non-zero cut-off radius around the reference point not only makes the asymptotic distribution Gaussian, but also changes the scaling variable from $x \sim n/\sqrt{t}$ to $x \sim n/\left(t^{1/4} \ln^{1/2} t\right)$. The reason for such a behaviour is the qualitatively different flux dependence of the 'Lifshitz tail' at $R = 0$ and $R \neq 0$ (compare (38) with (43) or (51)).

The boundary conditions for a walker at the surface of the removed disc turn out to be irrelevant for the asymptotic form of $\mathcal{P}(n,t)$. It is interesting that the same situation takes place for a random walk without traps, in which case the asymptotic distribution has the form (3) (with different numerical factors in the definition of the scaling variable x though) both for reflecting and absorbing boundaries [7]. One can also speculate that the geometrical shape of the removed region becomes irrelevant at $t \to \infty$, so that the distribution (4) is actually asymptotically universal.

Concerning the results known for some other similar systems, one should mention the numerical simulations in [7] of a random walk on a lattice with random bonds. The winding probability distribution for such a system is argued to be Gaussian with the scaling variable $x \sim n/\sqrt{\ln t}$, being thus very similar to the self-avoiding case [8, 9].

In conclusion, the theory of random walks with various topological constraints in disordered media seems to be an interesting and rich field for theoretical investigations. Apart from apparent practical relevance for the physics of polymers or vortex lines, this theory also has intimate connections with many other areas of statistical physics. Among some interesting open problems here, let us just mention the calculation of the winding angle distribution for a Brownian particle diffusing in a medium with critical disorder (for instance, in a quenched random velocity field [25]), or the vortex entanglement in a superconductor with point defects, etc.

Acknowledgments

I would like to thank D E Khmelnitskii for stimulating discussions. The work was financially supported by the EPSRC (grant no RG 22473).

Appendix

In this appendix we derive the small-x asymptotics of the expansions of the Bessel functions with respect to an index. Let

$$J_\nu(x) = \sum_{m=0}^{\infty} j_m(x)\, \nu^m \qquad Y_\nu(x) = \sum_{m=0}^{\infty} y_m(x)\, \nu^m \tag{A1}$$

$(j_0(x) = J_0(x),\ y_0(x) = Y_0(x))$. It follows from (36) that

$$j_1(x) = \tfrac{1}{2}\pi\, Y_0(x) \simeq \ln x \qquad y_1(x) = -\tfrac{1}{2}\pi\, J_0(x) \simeq -\tfrac{1}{2}\pi + \mathrm{O}\left(x^2\right).$$

From the integral representation of the Bessel functions [24]

$$J_\nu(x) = -\frac{\sin\nu\pi}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-x\sinh t-\nu t} + \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \cos(x\sin\theta - \nu\theta)$$

$$Y_\nu(x) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \left(\mathrm{e}^{\nu t} + \mathrm{e}^{-\nu t}\cos\nu\pi\right)\mathrm{e}^{-x\sinh t} + \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \sin(x\sin\theta - \nu\theta)$$

we obtain

$$j_2(x) = \int_0^\infty \mathrm{d}t\, t\mathrm{e}^{-x\sinh t} - \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, \theta^2\cos(x\sin\theta)$$

$$y_2(x) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \left(t^2 - \tfrac{1}{2}\pi^2\right)\mathrm{e}^{-x\sinh t} - \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, \theta^2\sin(x\sin\theta).$$

The dominant contributions to these expressions at small x come from the first integrals on the right-hand sides, whose asymptotics can be easily evaluated, resulting in

$$j_2(x) \simeq \frac{1}{2}\ln^2 x \qquad y_2(x) \simeq \frac{1}{3\pi}\ln^3 x. \tag{A2}$$

References

[1] Wiegel F W 1986 *Introduction to Path-integral Methods in Physics and Polymer Science* (Singapore: World Scientific)

[2] For a review, see Blatter G, Feigelman M V, Geshkenbein V B, Larkin A I and Vinokur V M 1994 *Rev. Mod. Phys.* **66** 1125

[3] Spitzer F 1958 *Trans. Am. Math. Soc.* **87** 187

[4] Edwards S F 1967 *Proc. Phys. Soc.* **91** 513

[5] Comtet A, Desbois J and Monthus C 1993 *J. Stat. Phys.* **73** 433

[6] Saleur H 1994 *Phys. Rev.* E **50** 1123

[7] Drossel B and Kardar M 1996 *Phys. Rev.* E **53** 5861

[8] Fisher M E, Privman V and Redner S 1984 *J. Phys. A: Math. Gen.* **17** L569

[9] Duplantier B and Saleur H 1988 *Phys. Rev. Lett.* **60** 2343

[10] Khandekar D C and Wiegel F W 1988 *J. Phys. A: Math. Gen.* **21** L563

[11] Duplantier B 1989 *J. Phys. A: Math. Gen.* **22** 3033

[12] For a review, see Grosberg A and Nechaev S 1993 *Adv. Polymer Sci.* **106** 1

[13] Balagurov B Ya and Vaks V G 1974 *Sov. Phys.–JETP* **38** 968

[14] Grassberger P and Procaccia I 1982 *J. Chem. Phys.* **77** 6281

[15] Nelson D R and Vinokur V M 1992 *Phys. Rev. Lett.* **68** 2398

[16] Samokhin K V 1998 *J. Phys. A: Math. Gen.* **31** 8789

[17] Rudnik J and Hu Y 1987 *J. Phys. A: Math. Gen.* **20** 4421

[18] Belisle C 1989 *Ann. Probab.* **17** 1377

[19] Lifshitz I M 1965 *Sov. Phys.–Usp.* **7** 549

[20] Friedberg R and Luttinger J M 1975 *Phys. Rev.* B **12** 4460

[21] Lubensky T C 1984 *Phys. Rev.* A **30** 2657

[22] Langer J S 1967 *Ann. Phys.* **41** 108

[23] Callan C and Coleman S 1977 *Phys. Rev.* D **16** 1762

[24] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)

[25] Samokhin K V unpublished